Spectral sequences

Let $\mathcal{A}$ be an abelian category.

Def: A (cohomology) spectral sequence in $\mathcal{A}$ is

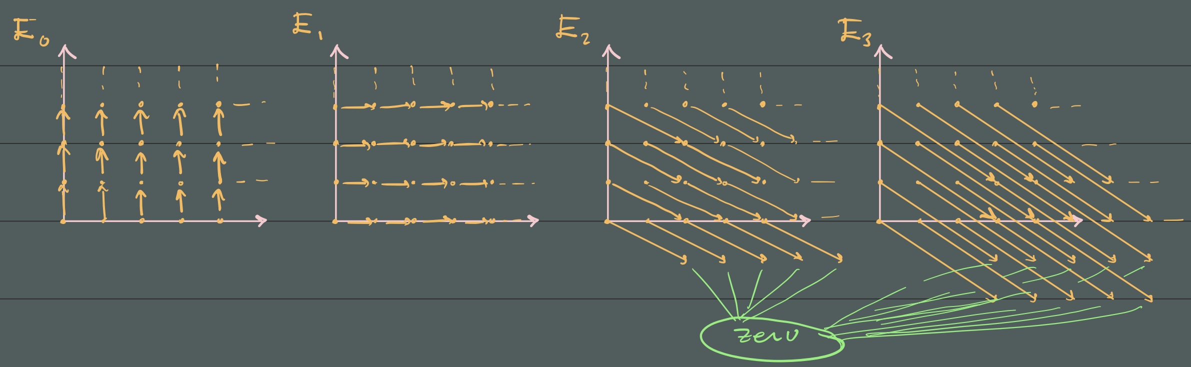
(1) a collection of obj's. $\{E_r^{pq}\} \subseteq \mathrm{Ob}(\mathcal{A})$ where $r \in \mathbb{Z}_{\geq 0}$, $p,q \in \mathbb{Z}$,

(2) morphisms $d_r^{pq}: E_r^{pq} \longrightarrow E_r^{p+r,\,q-r+1}$ s.t. $d \circ d = 0$,

(3) isomorphisms

$$E_{r+1}^{pq} \cong \ker(d_r^{pq}) / \mathrm{im}(d_r^{p-r,\,q+r-1}).$$

Intuition: Think of a spectral sequence as a book where the r:th page is the "grid" E_r^{pq} with its differentials and the (r+1):th page is the "cohomology of the r:th page." (first quadrant)



Rmk: There is also a homological version, where where all arrows are reversed.

Bounded convergence: A spectral seq. is bounded if for each $n \in \mathbb{Z}$, there are only finitely many non-zero E_0^{pq} s.t. $p+q=n$. If this happens then there exists an r_0 s.t. $E_r^{pq} \cong E_{r+1}^{pq}$ for all $r \geq r_0$ and we denote this stable value by E_∞^{pq}.

Def: we say that E_r^{pq} converges to H^* if there is a finite filtration

$$0 = F^t H^n \subseteq \cdots \subseteq F^{p+1} H^n \subseteq F^p H^n \subseteq \cdots \subseteq F^s H^n = H^n$$

s.t. $E_\infty^{pq} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$. This is written as

$$\boxed{E_a^{pq} \Rightarrow H^{p+q}}$$

Spectral sequence of a filtration

Whenever we have a bounded (+ finite) (= finite in each degree) filtration $\cdots \subseteq F_{p-1} C \subseteq F_p C \subseteq \cdots$ we get a convergent spectral seq.

$$E_1^{pq} = H^{p+q}(F_p C / F_{p-1} C) \Rightarrow H^{p+q}(C)$$

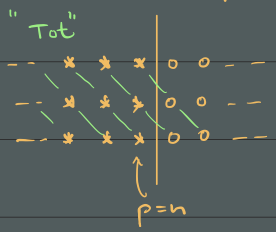
(see Wiebel 5.4.6 and 5.5.1).

This enables us to define the spectral sequence of a double complex:

Spectral seq. of a double complex:

If we have a double complex $C^{\bullet\bullet}$ we may filter the total complex $\mathrm{Tot}(C^{\bullet\bullet})$ by the columns of $C^{\bullet\bullet}$, i.e., we take ${}^{I}F_n \mathrm{Tot}(C^{\bullet\bullet})$ to be the total complex of

${}^{I}\tau_{\leq n} C$:



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If $C^{\bullet\bullet}$ lives in the first quadrant, then this filtration is bounded and we get a convergent spectral sequence

$$\boxed{{}^{I}E_2^{pq} = H_h^p H_v^q (C^{\bullet\bullet}) \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet\bullet})).}$$

horizontal vertical

There is also a filtration by rows which yields a conv. spectral seq.

$$\boxed{{}^{II}E_2^{pq} = H_v^p H_h^q (C^{\bullet\bullet}) \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet\bullet})).}$$

This enables us to define the spectral sequence of a composition of left exact functors:

The Grothendieck spectral sequence

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories and assume that $\mathcal{A}$ and $\mathcal{B}$ has enough injectives. Suppose that we have left exact functors

$$\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C}$$

such that G sends injective objects to F-acyclic objects. (yields $RF \circ RG \simeq R(FG)$)

Theorem: With the above setup, there is a convergent first quadrant (cohom.) spectral sequence for each $A \in \mathcal{A}$:

$${}^{I}E_2^{pq} = (R^pF)(R^qG)(A) \Longrightarrow R^{p+q}(FG)(A).$$

proof (sketch): Choose an inj. resolution $A \to I$ and apply G to get a complex $G(I)$ in $\mathcal{B}$. Take a Cartan-Eilenberg resolution $C^{\bullet\bullet}$ of $G(I)$:

$$\begin{tikzcd} \vdots & \vdots & \\ C^{01} \arrow[r] \arrow[u] & C^{11} \arrow[r] \arrow[u] & \cdots \\ C^{00} \arrow[r] \arrow[u] & C^{10} \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

(exists by Weibel's 5.7.2)

Now apply F:

If we filter by columns we get

$${}^{I}E_2^{pq} = H^p((R^qF)(GI)) \Longrightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet\bullet})).$$

Since each $G(I)^n$ is F-acyclic, the E_2-page will take the form ($R^qF(GI^p) = 0$ if $q > 0$)

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
F(G(I^0)) & F(G(I^1)) & F(G(I^2)) & \cdots \\
 & & 0 & 0
\end{array}$$

so we get

$$R^p(FG)(A) \cong H^p(\mathrm{Tot}(C^{\bullet\bullet})).$$

On the other hand, filtering by rows, we get

$$ {}^{II}E_2^{pq} = (R^pF)\underbrace{H_h^q(G(I))}_{\cong R^qG(A)} \Longrightarrow R^{p+q}(FG)(A). \qquad \square$$

<u>Ex:</u> Suppose that we have a group G and a normal subgroup N:

$$1 \to N \to G \to G/N \to 1.$$

Then every G-module A restricts to an N-module and we have left exact functors

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{Z}G} \arrow[r, "(-)^N"] \arrow[rr, bend right, "(-)^G"'] & \mathrm{Mod}_{\mathbb{Z}(G/N)} \arrow[r, "(-)^{G/N}"] & \mathrm{Ab}
\end{tikzcd}$$

The Grothendieck spectral seq. looks like

$$\boxed{E_2^{pq} = H^p(G/N, H^q(N,A)) \Longrightarrow H^{p+q}(G,A)}$$

This is usually called the <u>Hochschild-Serre</u> <u>spectral sequence</u>.

<u>Example:</u>

The cohomology of D_8 and Q_8.

We computed some low dim. cohomology groups of D_8 and Q_8 last week using GAP. Now let's compute these cohomology groups by hand using HS-spectral sequences.

- Let G be either D_8 or Q_8.
- We have $D_8^{ab} \cong Q_8^{ab} \cong (\mathbb{Z}/2\mathbb{Z})^2$ and hence

$$H^1(G, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(G^{ab}, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

- Consider the HS-spectral sequence associated to the exact sequence $0 \to \mathbb{Z}/4\mathbb{Z} \to G \to \mathbb{Z}/2\mathbb{Z} \to 0$ where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}/4\mathbb{Z}$ by $1.x = -x$, where we consider the trivial G-module $\mathbb{Q}/\mathbb{Z}$:

  ($G = D_8$ or Q_8, $N = \mathbb{Z}/4\mathbb{Z}$, $G/N = \mathbb{Z}/2\mathbb{Z}$)

! First note that $H^1(N, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{gr}(N, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ acts by $1.\varphi = -\varphi$. !

The E_2-page takes the form:

$$\begin{array}{l|cccc}
 & 0 & 0 & 0 & 0 \\
 & (\mathbb{Z}/4\mathbb{Z})^{G/N} & H^1(G/N, \mathbb{Z}/4\mathbb{Z}) & * & * \\
 & \mathbb{Z} & H^1(G/N, \mathbb{Q}/\mathbb{Z}) & H^2(G/N, \mathbb{Q}/\mathbb{Z}) & H^3(G/N, \mathbb{Q}/\mathbb{Z})
\end{array}$$

The filtration on $E_\infty^2 = H^2(G, \mathbb{Q}/\mathbb{Z})$ takes the form

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & F^2E^2_\infty & \longrightarrow & F^2E^2_\infty/0 \cong E_\infty^{2,0} \cong \overset{0}{\overset{\shortparallel}{H^2(G/N, \mathbb{Q}/\mathbb{Z})}} \\
0 & \longrightarrow & \overset{0}{\overset{\shortparallel}{F^2E^2_\infty}} & \longrightarrow & F^1E^2_\infty & \longrightarrow & F^1E^2_\infty/F^2E^2_\infty \cong E_\infty^{1,1} \cong \ker\begin{pmatrix} H^1(G/N, \mathbb{Z}/4\mathbb{Z}) \\ \downarrow \\ H^3(G/N, \mathbb{Q}/\mathbb{Z}) \end{pmatrix} \\
0 & \longrightarrow & F^1E^2_\infty & \xrightarrow{\sim} & E^2_\infty & \longrightarrow & E^2_\infty/F^1E^2_\infty \cong E_\infty^{0,2} = 0 \\
 & & & & \shortparallel & & \\
 & & & & H^2(G, \mathbb{Q}/\mathbb{Z}) & &
\end{array}$$

This gives an exact sequence

$$\boxed{0 \longrightarrow H^2(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(G/N, \mathbb{Z}/4\mathbb{Z}) \longrightarrow H^3(G/N, \mathbb{Q}/\mathbb{Z})}$$

<u>Fact:</u> The differentials on the E_2-page (for a trivial module) are always given (up to sign) by "cup product" with the element

$$(0 \to N \to G \to G/N \to 0) \in H^2(G/N, N).$$

Since D_8 is a semi-direct product $D_8 \cong \mathbb{Z}/4\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, it represents zero in $H^2(G/N, N)$ and cup product with this element will be the zero map. Thus
$H^3(D_8, \mathbb{Z}) \cong H^2(D_8, \mathbb{Q}/\mathbb{Z}) \cong H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z}$.

On the other hand, Q_8 is <u>not</u> a semi-direct product and will give an isomorphism $H^1(G/N, H^1(N, \mathbb{Q}/\mathbb{Z})) \xrightarrow{\sim} H^3(G/N, \mathbb{Q}/\mathbb{Z})$ (enough to check that it is not zero). Thus

$$H^3(Q_8, \mathbb{Z}) \cong H^2(Q_8, \mathbb{Q}/\mathbb{Z}) = 0.$$

- Next we use the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{2} \mathbb{Q}/\mathbb{Z} \to 0$$

to get a long exact sequence

$$\cdots \to H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{2} H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G, \mathbb{Z}/2\mathbb{Z}) \to H^2(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{0} \cdots$$

to see that ($\mathbb{F}_2$-vector spaces)

$$H^2(G, \mathbb{Z}/2\mathbb{Z}) \cong H^1(G, \mathbb{Q}/\mathbb{Z}) \oplus H^2(G, \mathbb{Q}/\mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^3, & G = D_8 \\ (\mathbb{Z}/2\mathbb{Z})^2, & G = Q_8 \end{cases}$$